Topology and its Applications 49 (1993) 237-249
North-Holland

Strong shape of uniform spaces

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Received 19 August 1991
Revised 16 March 1992

Abstract

Segal, J., S. Spież and B. Günther, Strong shape of uniform spaces, Topology and its Applications 49 (1993) 237-249.

A strong shape category for finitistic uniform spaces is constructed and it is shown, that certain nice properties known from strong shape theory of compact Hausdorff spaces carry over to this setting. These properties include a characterization of the new category as localization of the homotopy category, the product property and obstruction theory.

Keywords: Uniform spaces, finitistic uniform spaces, strong shape, Cartesian products, localization, Samuel compactification, obstruction theory.

AMS (MOS) Subj. Class.: 54C56, 54E15, 55N05, 55P55, 55S35.

Introduction

In strong shape theory of topological spaces one encounters several instances, where the desired extension of theorems from compact spaces to more general ones either leads to difficult unsolved problems or is outright impossible. Examples are:

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* This paper was written while this author was visiting the University of Washington.

** This paper was written while this author was visiting the University of Washington, and this visit was supported by a DFG fellowship.

Can the strong shape category of topological spaces **ssh(Top)** be represented in the homotopy category, i.e., does there exist a functor T: **ssh(Top)** → **HTop** such that the equation **ssh(Top)**(X, Y) = **HTop**$(T(X), T(Y))$ holds? Does the Cartesian product $X \times Y$ possess the categorical product property in **ssh(Top)** for any two spaces X and Y? Are there well-behaved compactifications? Observe that the Stone–Čech compactification does not even factor over the homotopy category. In this paper we attempt to show that these questions are posed in the wrong context; they have affirmative answers if one uses uniform spaces instead of topological ones.

1. The strong shape category of uniform spaces

Spaces and maps are always understood in the uniform sense, unless something else is explicitly stated. Uniform spaces are not required to be Hausdorff.

Definition 1.1. Let X be an arbitrary uniform space and Y a metrizable uniform space. The *semi-uniform product* $X * Y$ is the uniform space with underlying set $X \times Y$, and the stacked coverings $\{U \times V \mid U \in \mathcal{U},\ V \in \mathcal{V}_U\}$ as base of uniform coverings. Here $\mathcal{U}$ is a uniform covering of X and to each $U \in \mathcal{U}$ is assigned a uniform covering $\mathcal{V}_U$ of Y [11, Theorem III.28].

We observe that $X * Y$ does not depend symmetrically on X and Y, and that a map $X * Y \to Z$ is uniformly continuous if and only if the adjoint map $X \to U(Y, Z)$ is uniformly continuous. $U(Y, Z)$ denotes the set of uniformly continuous maps $Y \to Z$, endowed with the structure of uniform convergence [11, Theorem III.26]. $X * Y$ carries a uniformity finer than the usual product uniformity, in general it is strictly finer.

Definition 1.2. Two uniformly continuous maps $f, g: X \to Y$ are *homotopic* if there is a uniformly continuous map $X * I \to Y$ with $H_0 = f$ and $H_1 = g$. A *fibration* is a uniformly continuous map $\pi: E \to B$ admitting solutions of the following relative lifting problem:

$$\begin{tikzcd} X * \{0\} \approx X \arrow[r] \arrow[d, hook] & E \arrow[d, "\pi"] \\ X * I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd} \tag{1}$$

The relation of homotopy is a congruence and hence gives rise to a homotopy category **HUnif** of uniform spaces. Associativity of the homotopy relation follows from Lemma 1.3:

Lemma 1.3. *Let a space X be covered by finitely many sets $A_0, \ldots A_n$, and let $f: X \to Y$ be a map, whose restriction to each A_i is uniformly continuous. If each uniform covering*

$\mathcal{U}$ of X admits a uniform refinement $\mathcal{V}$, such that for any two $\mathcal{V}$-near points $x \in A_i$, $x' \in A_k$ there is $x'' \in A_i \cap A_k$ with x and x'' respectively x' and x'' $\mathcal{U}$-near, then $f: X \to Y$ is uniformly continuous.

We emphasize that the sets A_i mentioned above need not be open or closed.

By **pro-Unif** we denote the category of inverse systems of uniform spaces; the homotopy relation of Definition 1.2 extends to the pro-category in an obvious way and leads to an elementary homotopy category π(**pro-Unif**), which is an intermediate step in the construction of the Steenrod homotopy category **ho**(**pro-Unif**).

Definition 1.4. A morphism $\boldsymbol{i}: \boldsymbol{A} \to \boldsymbol{X}$ is a *trivial cofibration* if it has the left lifting property with respect to all fibrations of uniform spaces. An inverse system of $\boldsymbol{Z}$ uniform spaces is *fibrant* if for every trivial cofibration $\boldsymbol{i}: \boldsymbol{A} \to \boldsymbol{X}$ and every map $\boldsymbol{f}: \boldsymbol{A} \to \boldsymbol{Z}$ there is $\boldsymbol{g}: \boldsymbol{X} \to \boldsymbol{Z}$ with $\boldsymbol{gi} = \boldsymbol{f}$ [3, Definitions 1.3 and 1.6].

Proposition 1.5. *The full subcategory* π(**pro-Unif**)$_{\mathrm{f}} \subseteq \pi$(**pro-Unif**) *spanned by all fibrant inverse systems is reflexive.*

Proof. We consider an inverse system $\boldsymbol{X}$ of uniform spaces over a cofinite index set and perform the construction described in [3, Section 2] leading to an inverse system $\hat{\boldsymbol{X}}$ of uniform spaces over the same index set and a level system of embeddings $i_\lambda : X_\lambda \to X_\lambda$, such that $i_\lambda(X_\lambda)$ is a uniform strong deformation retract of $\hat{X}_\lambda$, and such that the following property holds:

(i) For every index λ the map $\lim_{\mu<\lambda} \hat{p}^\lambda_\mu : \hat{X}_\lambda \to \lim_{\mu<\lambda} \hat{X}_\mu$ is a fibration.

$\hat{p}^\lambda_\mu : \hat{X}_\lambda \to \hat{X}_\mu$ are the bonding maps of $\hat{\boldsymbol{X}}$. It follows immediately that $\hat{\boldsymbol{X}}$ is fibrant, and we claim that our level morphism $\boldsymbol{i}: \boldsymbol{X} \to \hat{\boldsymbol{X}}$ can be factored $\boldsymbol{i} = \boldsymbol{qj}$ with $\boldsymbol{q}$ a homotopy equivalence in π(**pro-Unif**) and $\boldsymbol{j}$ a trivial cofibration. This factorization is furnished by the usual mapping cylinder construction; this is possible because as a consequence of [11, Exercise 7, p. 11] the category of uniform spaces contains all pushout diagrams and hence mapping cylinders. $\square$

Proposition 1.5 enables us to define the *Steenrod homotopy category* **ho**(**pro-Unif**) by inverting all trivial cofibrations in π(**pro-Unif**); alternatively **ho**(**pro-Unif**) is the full image of a reflecting functor $R: \pi(\textbf{pro-Unif}) \to \pi(\textbf{pro-Unif})_{\mathrm{f}}$.

Definition 1.6. A Hausdorff uniform space P is an *ANRU-space*, if every uniformly continuous map defined on a (not necessarily closed) subspace A of a uniform space X and taking values in P has a uniformly continuous extension over a uniform neighborhood of A in X (cf. [11, p. 82]). A uniform space X is *admissible* if it admits a **ho**(**pro-ANRU**)-reflection in **ho**(**pro-Unif**).

By **ho(pro-ANRU)** $\subseteq$ **ho(pro-Unif)** we mean the full subcategory of all inverse systems of ANRU-spaces. Later we will see that for such systems the fibrant reflector constructed above consists of ANRU-spaces and consequently **ho(pro-ANRU)** is obtained from π(**pro-ANRU**) by inverting the trivial cofibrations there, thereby removing the ambiguity in notation.

The analogy between the topological and the uniform development of strong shape breaks down at this point: We are not able to show that a given uniform space is admissible unless it satisfies the following finiteness assumption:

Definition 1.7. A uniform space X is *finitistic* if every uniform covering of X has a uniform refinement of finite order (cf. [19, Theorem 3.1] for the topological case).

Examples. (a) Every compact space with its unique uniformity is finitistic.

(b) Every subspace of a Euclidean space with the uniformity induced by the Euclidean metric is finitistic.

(c) Every paracompact Hausdorff space of finite covering dimension with the finest uniform structure compatible with the topology is finitistic.

(d) There is a nice characterization of finitistic topological spaces, see [10]: A paracompact Hausdorff topological space is finitistic if and only if it contains a compact subspace K, such that every closed subspace disjoint from K has finite covering dimension. But there are finitistic uniform spaces which do not satisfy this condition, e.g. $\mathbb{Z} \times Q$, the product of the discrete space of integers and the Hilbert cube with the usual product structure.

Resolutions are defined as in the topological case, namely by conditions (R1) and (R2) in [15, p. 74] with all spaces, maps and coverings uniform. We also have to consider conditions (B1) and (B2) from [15, p. 76], translated to the uniform language. (B1) reads as follows:

(B1) For every index λ and every uniform neighborhood U of $p_\lambda(X)$ in X_λ there is $\mu \geqslant \lambda$ with $p_\lambda^\mu(X_\mu) \subseteq U$.

The implications (R2)$\Leftrightarrow$(B1) and (R1)$\Rightarrow$(B2) are true in all cases, and (B1) $\wedge$ (B2)$\Rightarrow$(R1) holds if all X_λ are finitistic. To prove this one applies uniform counterparts of topological theorems: By [11, Theorem II.21] every Hausdorff uniform space may be considered as a bounded subspace of a locally convex vector space, and by [11, Theorem IV.11] every uniform covering of an arbitrary uniform space has a subordinated equiuniformly continuous partition of unity endowed with the same index set. A significant difference between the topological and the uniform case occurs, when one has to consider maps of the form $f(x) = \Sigma_\iota \varphi_\iota(x) y_\iota$, where $\{\varphi_\iota\}$ is an equiuniformly continuous partition of unity and $\{y_\iota\}$ a bounded subset of some locally convex vector space. Uniform continuity of f can be shown only if the partition of unity is of finite order, say N: If p is a continuous seminorm on the locally convex space and r a bound for all $p(y_\iota)$, then we can estimate $p(f(x) - f(x')) \leqslant 2rN \sup_\iota |\varphi_\iota(x) - \varphi_\iota(x')|$.

Every polyhedron (with a given triangulation) carries a natural uniform structure (cf. [11, p. 58], "uniform complex").

Theorem 1.8. *Every finitistic uniform space has a resolution in finite dimensional polyhedra.*

The proof follows the pattern of [15, Theorem I.6.7]; in the final step one observes that by [11, Theorem IV.6] every subset of a finite dimensional polyhedron has a base of uniform neighborhoods consisting of polyhedra. The inverse system obtained satisfies (B1) and (B2) and due to the finiteness assumption it is a resolution.

From Theorem 1.8 and [11, Theorem V.15] we conclude:

Corollary 1.9. *Every finitistic uniform space has a resolution in ANRU-spaces.*

Theorem 1.10. *Every ANRU-resolution induces a* **ho(pro-ANRU)**-*reflection. In particular every finitistic uniform space is admissible.*

The proof can be taken from [3, Section 3] if one pays attention to the following modifications:

(ii) A uniform *cofibration* is defined to be an inclusion map $A \hookrightarrow X$ such that the inclusion $X * \{0\} \cup A * I \hookrightarrow X * I$ has the left lifting property with respect to all fibrations of uniform spaces.

In uniform context this condition seems to be more restrictive than the usual definition of cofibrations, but it is fulfilled by the inclusion maps of bottom and top of a mapping cylinder. If $A \hookrightarrow X$ is a uniform cofibration then so is $X * \dot{I} \cup A * I \hookrightarrow X * I$; we observe that by [11, Theorem III.25 and Corollary III.29] $X * \dot{I}$ and $A * I$ are subspaces of $X * I$.

(iii) For $\boldsymbol{X} \in \pi(\mathbf{pro\text{-}ANRU})$ the reflector $\hat{\boldsymbol{X}} \in \pi(\mathbf{pro\text{-}Unif})_{\mathrm{f}}$ of Proposition 1.5 may be chosen from the subcategory $\pi(\mathbf{pro\text{-}ANRU})$.
For proof one can follow [3, Section 2]. $\hat{\boldsymbol{X}}_\lambda$ is a space of functions defined on a finite complex, and so it is built in finitely many steps by pullback constructions of the following form, where P and Q are ANRU-spaces:

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & P^{I^n} \arrow[d] \\ Q \arrow[r] & P^{\dot{I}^n} \end{tikzcd} \tag{2}$$

Taking adjoints and observing [11, Theorem III.25 and Corollary III.29] it is seen that Z is an ANRU-space too. (We are not able to show that Z is an ANRU-space when the right vertical arrow in the pullback diagram (2) is replaced by an arbitrary fibration of ANRU-spaces, since in uniform context fibrations fail to be regular.)

Definition 1.11. For each admissible uniform space X we choose a reflector $\boldsymbol{R}(X) \in$ **ho(pro-ANRU)**. The *strong shape category* **ssh** of uniform spaces is defined to have all admissible spaces as objects, whereas the morphisms are given by $\mathbf{ssh}(X, Y) := \mathbf{ho}(\mathbf{pro}\text{-}\mathbf{ANRU})(\boldsymbol{R}(X), \boldsymbol{R}(Y))$. The *strong shape functor* $\eta : \mathbf{HaUnif} \to \mathbf{ssh}$ from the homotopy category of admissible uniform spaces to the strong shape category is induced by $\boldsymbol{R}$.

Remark. Various authors [4-6, 16] have considered *ordinary* shape categories for uniform metric spaces using an approach analogous to Borsuk's shape for compacta. Applications of *strong* shape to uniform spaces seem to be new.

Theorem 1.12. *The strong shape functor* $\eta : \mathbf{HaUnif} \to \mathbf{ssh}$ *has a right adjoint* $\boldsymbol{T} : \mathbf{ssh} \to \mathbf{HUnif}$.

Proof. For every admissible space X we choose a *fibrant* $\mathbf{ho}(\mathbf{pro}\text{-}\mathbf{ANRU})$-reflector $\boldsymbol{R}(X)$. Then for any two admissible spaces X, Y we obtain the following chain of natural bijections:

$$\mathbf{ssh}(Y, X) = \mathbf{ho}(\mathbf{pro}\text{-}\mathbf{ANRU})(\boldsymbol{R}(Y), \boldsymbol{R}(X)) \approx \mathbf{ho}(\mathbf{pro}\text{-}\mathbf{Unif})(Y, \boldsymbol{R}(X))$$

$$= \pi(\mathbf{pro}\text{-}\mathbf{Unif})(Y, \boldsymbol{R}(X)) \approx \mathbf{HUnif}\left(Y, \varprojlim \boldsymbol{R}(X)\right).$$

Therefore we can define our right adjoint functor by $T(X) := \varprojlim \boldsymbol{R}(X)$. $\square$

Remark. We do not know whether $T(X)$ is admissible, but in Section 2 we will see that this holds at least for finitistic spaces X. We observe that in this case the classifying space $T(X)$ is derived from a cofinite ANRU-resolution $\{h_\lambda\} : X \to \{p_\mu^\lambda : Y_\lambda \to Y_\mu \mid \lambda \geq \mu \in \Lambda\}$ of X via the construction in [3, Section 2]: Let $K(\Lambda)$ denote the complex of the ordered set Λ and $K(\Lambda, \lambda)$ the full subcomplex determined by all indices $\mu \geq \lambda$; these spaces are *topological spaces* carrying the Whitehead topology. Let $C(K(\Lambda, \lambda), Y_\lambda)$ denote the space of all *continuous* functions between the indicated spaces carrying the *uniform structure* of uniform convergence on compact subsets. Then $T(X)$ is the subset

$$T(X) \subseteq \prod_{\lambda \in \Lambda} C(K(\Lambda, \lambda), Y_\lambda)$$

formed by all families of continuous functions $\omega_\lambda : K(\Lambda, \lambda) \to Y_\lambda$, such that for $\lambda \geq \mu$, ω_μ is the restriction of $p_\mu^\lambda \omega_\lambda$ (cf. [14, Section I.10]). The product carries the usual product uniformity.

2. Peculiar properties of the uniform case

Definition 2.1. An inclusion map $A \hookrightarrow X$ of uniform spaces is called *SSDR-map*, if for any fibration $\pi : E \to B$ of ANRU-spaces the following lifting problem is solvable:

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & E \arrow[d, "\pi"] \\ X \arrow[r] \arrow[ur, dashed] & B \end{tikzcd} \tag{3}$$

A uniform space Z is *fibered*, if for every SSDR-map $A \hookrightarrow X$ and every map $f: A \to Z$ there is an extension of f over X. By **HfUnif** $\subseteq$ **HaUnif** we denote the full subcategory of all uniform spaces having the strong shape of a finitistic uniform space; **fssh** $\subseteq$ **ssh** is the full subcategory with the same objects as **HfUnif**.

Our definition of SSDR-maps is condition (3) of [2, Theorem 1.2], which seems to be different from (2) when uniform spaces are considered. The definition of fibered spaces is from [2, Definition 2.1].

Proposition 2.2. (a) *For every admissible space X the classifying space $T(X)$ (cf. Theorem 1.12) is fibered.*

(b) *Every SSDR-map is invertible in* **ssh**.

Proof. For (a) it suffices to observe that in the proof of Theorem 1.12 $T(X)$ was constructed as limit of a cofinite inverse system, such that for all indices λ the map $\lim_{\mu<\lambda} \hat{p}^{\lambda}_{\mu} : \hat{Z}_{\mu} \to \lim_{\mu<\lambda} \hat{Z}_{\lambda}$ is a fibration of ANRU-spaces. (b) follows from (a). □

Lemma 2.3. *We consider a resolution $\{h_\lambda\}: X \to \{p^{\lambda}_{\mu} : Y_\lambda \to Y_\mu \mid \lambda \geq \mu \in \Lambda\}$ of some uniform space X and set $h := \lim h_\lambda : X \to X^* := \lim Y_\lambda$.*

(a) *$h(X)$ is dense in X^*.*

(b) *h is a strong shape equivalence.*

Proof. (a) When $y \in X^*$ is not in the closure of $h(X)$ there must be a uniform covering $\mathcal{U}$ of X^* with $y \notin \mathrm{St}(h(X), \mathcal{U})$. From the definition of the uniform structure of an inverse limit we conclude the existence of an index λ and a uniform covering $\mathcal{V}$ of Y_λ, such that $p_\lambda^{-1}(\mathcal{V})$ refines $\mathcal{U}$ and in particular $p_\lambda(y) \notin \mathrm{St}(h_\lambda(X), \mathcal{V})$. By condition (B1) there is $\mu \geq \lambda$ with $p^{\mu}_{\lambda}(Y_\mu) \subseteq \mathrm{St}(h_\lambda(X), \mathcal{V})$, which is a contradiction.

(b) It suffices to show that $h: X \to X^*$ is a resolution, where X^* is considered as rudimentary inverse system, then the uniform analog of [3, Proposition 3.4] leads to the desired result. Condition (R1) is clear, and (B1) follows from (a), because X^* is the only uniform neighborhood of $h(X)$. □

Remark. If all the spaces Y_λ are complete Hausdorff uniform spaces, then $h: X \to X^*$ equals the natural map of X into its Hausdorff completion [1, Ch.II, § 3, no. 7, Theorem 3].

Proposition 2.4. *For every admissible space X its Hausdorff completion $\tilde{X}$ is admissible too and the natural map $X \to \tilde{X}$ is a strong shape equivalence; if X is Hausdorff it is an SSDR-map. In particular every admissible space has the strong shape of a complete Hausdorff space.*

Proof. Since by [11, Theorem V.14] every ANRU-space is a complete Hausdorff space diagram (3) has a unique filler, if the left vertical arrow equals $\varphi : X \to \tilde{X}$. If X is Hausdorff then φ is an inclusion and therefore an SSDR-map, in general this lifting property still suffices to show that φ is a strong shape equivalence. □

The following theorem is known to hold in the class of topological spaces having the strong shape of a compact Hausdorff space (see [9]), but for general topological spaces the problem is unsolved. Our theorem and the following two corollaries indicate that uniform spaces provide a more adequate framework for this type of question.

Theorem 2.5. *The strong shape functor η :* **HfUnif** *→* **fssh** *localizes* **HfUnif** *at the class of all SSDR-maps. The image of its right adjoint functor T :* **fssh** *→* **HUnif** *is contained in* **HfUnif**, *and T :* **fssh** *→* **HfUnif** *is a fully faithful embedding.*

Proof. For every space $X \in$ **HfUnif** let $f_X : X \to T(X)$ be the map, which corresponds to the identity under the adjunction isomorphism **ssh**$(X, X) \approx$ **HfUnif**$(X, T(X))$. By [7, Proposition I.1.3] it suffices to show that each $\eta(f_X)$ is a strong shape equivalence, and clearly we can restrict our attention to finitistic uniform spaces X. Then as $T(X)$ we can use the function space described in the remark following the proof of Theorem 1.12, and as $f_X : X \to T(X)$ the map assigning to a point $x \in X$ the family of constant functions $\omega_\lambda : K(\Lambda, \lambda) \to Y_\lambda$, $\omega_\lambda \equiv h_\lambda(x)$. If $X^* \subseteq T(X)$ denotes the subspace of all constant functions, then f_X equals the composition of $\lim h_\lambda : X \to \lim Y_\lambda \approx X^*$ and the inclusion map $i : X^* \hookrightarrow T(X)$. By Lemma 2.3 it suffices to show that i is a strong shape equivalence, and therefore we need to apply the techniques of [9, Lemma 1.5(b) and Theorem 1.6] to the uniform case. The definition of the uniform structure of $T(X)$ in the above mentioned remark provides us with a base of uniform neighborhoods of X^* in $T(X)$ consisting of all sets $U(C, \lambda, \mathcal{U})$ of the following form:

(iv) Let $C \subseteq K(\Lambda)$ be a finite subcomplex, all of whose vertices μ are bounded by an index λ, and let $\mathcal{U}$ be an entourage of Y_λ. Then $U(C, \lambda, \mathcal{U}) \subseteq T(X)$ consists of all families of functions $\omega_\nu : K(\Lambda, \nu) \to Y_\nu$, such that $\omega_{\lambda|C}$ lies within a $\mathcal{U}$-neighborhood of a constant function.

This means that the uniform analog of [9, Lemma 1.5(b)] holds, and as in step (b) of the proof of [9, Theorem 1.6] to a given uniform neighborhood U of X^* in $T(X)$ one can construct a deformation $D : T(X) * I \to T(X)$ stationary on X^* with $D_0 = \mathrm{id}$ and $D_1(T(X)) \subseteq U$. We conclude that the inclusion map of X^* into the bottom of the mapping cylinder of $X^* \hookrightarrow T(X)$ satisfies the uniform analog of condition (2), [2, Theorem 1.2], and since it is a cofibration it is an SSDR-map. □

Corollary 2.6. *Each finitistic space X is "improved" by $T(X)$, i.e., these two spaces have the same strong shape and for any admissible space Y, $\eta : \mathbf{HUnif}(Y, T(X)) \to \mathbf{ssh}(Y, T(X))$ is bijective.*

Proof. For any map $g : Y \to T(X)$ the naturality of the adjunction isomorphism provides us with the following commutative diagram:

$$\begin{array}{ccc} \mathbf{ssh}(T(X), T(X)) & \approx & \mathbf{HUnif}(T(X), T^2(X)) \\ \Big\downarrow{\scriptstyle \eta(g)^*} & & \Big\downarrow{\scriptstyle g^*} \\ \mathbf{ssh}(Y, T(X)) & \approx & \mathbf{HUnif}(Y, T^2(X)) \end{array} \tag{4}$$

Since the identity strong shape map is mapped to $f_{T(X)} : T(X) \to T^2(X)$ by the upper horizontal isomorphism we see that the composition of $\eta : \mathbf{HUnif}(Y, T(X)) \to \mathbf{ssh}(Y, T(X))$ with the adjunction isomorphism $\mathbf{ssh}(Y, T(X)) \approx \mathbf{HUnif}(Y, T^2(X))$ is induced by $f_{T(X)}$. We have already seen that this map is a strong shape equivalence, and it is even a homotopy equivalence, because the spaces involved are fibered. □

Remark. We do not know when $T(X)$ can be chosen to be finitistic. This is the reason for the somewhat awkward choice of the class **HfUnif**.

Theorem 1.12 holds in the topological context too, and the right adjoint functor gives a faithful and conservative representation of the topological strong shape category, see [8, Theorem 1.9]. This representation is full if and only if the topological strong shape category can be obtained from **HTop** by localization, but whether either of these properties holds is unknown. We can shed some light on this problem, because every paracompact Hausdorff topological space can be considered as a uniform space carrying the finest compatible uniformity, i.e., its uniform coverings are precisely all open coverings. This uniform space is finitistic if and only if the original topological space is topologically finitistic, and the uniform strong shape morphisms between two such spaces can be seen to coincide with the topological strong shape morphisms. Therefore Theorem 2.5 implies:

Corollary 2.7. *The topological strong shape category of finitistic spaces can be represented in the homotopy category of uniform spaces.*

We believe that Corollary 2.7 is the most natural solution to the representation problem. To solve the original problem of representation in **HTop** we would have to deal with the following question: If X and Y are finitistic, paracompact Hausdorff *topological* spaces and $T(X)$, $T(Y)$ the representing *uniform* spaces, is then every continuous map $T(X)\to T(Y)$ homotopic to a uniformly continuous map?

We now turn back to orthodox uniform considerations:

Theorem 2.8. *For any two finitistic uniform spaces X, Y the Cartesian product $X\times Y$ (carrying the usual product uniformity) is the categorical product in* **ssh**, *i.e., for every admissible space Z the assignment*

$$\mathbf{ssh}(Z, X\times Y)\to \mathbf{ssh}(Z, X)\times \mathbf{ssh}(Z, Y)$$

induced by the projection maps is bijective.

Proof. We take two resolutions $\{p_\lambda\}: X\to\{p_\lambda^{\lambda'}: X_{\lambda'}\to X_\lambda \mid \lambda'\geq\lambda\in\Lambda\}$ and $\{q_\mu\}: Y\to \{q_\mu^{\mu'}: Y_{\mu'}\to Y_\mu \mid \mu'\geq\mu\in M\}$ in finite dimensional polyhedra. Then the product map $\{p_\lambda\times q_\mu\}: X\times Y\to\{p_\lambda^{\lambda'}\times q_\mu^{\mu'}: X_{\lambda'}\times Y_{\mu'}\to X_\lambda\times Y_\mu \mid \lambda'\geq\lambda\in\Lambda,\ \mu'\geq\mu\in M\}$ can be shown to be a resolution by checking conditions (B1) and (B2). Constructing the space $T(X\times Y)$ from this resolution we see that the natural map $T(X\times Y)\to T(X)\times T(Y)$ is a homotopy equivalence. □

3. Extension and classification theorems

In this section we use a concept of homotopy different from Definition 1.2: *Two uniformly continuous maps $X\to Y$ are considered homotopic if and only if they can be connected by a uniformly continuous map $X\times I\to Y$ with $X\times I$ carrying the usual product uniformity.*

By $\mathcal{S}_m$ we denote the class of all m-connected compact polyhedra and by $\mathcal{S}_m^2$ the class of all pairs of m-connected compact polyhedra.

We say that the *deformation dimension of a uniform space X with respect to $\mathcal{S}_m$ is* $\leq n$ $(\operatorname{defdim}(X;\mathcal{S}_m)\leq n)$ if any uniformly continuous mapping $f: X\to P\in\mathcal{S}_m$ is uniformly homotopic to one whose image is contained in the n-skeleton $P^{(n)}$. Similarly, the *deformation dimension of a pair (X, A) of uniform spaces with respect to $\mathcal{S}_m^2$ is* $\leq n$ $(\operatorname{defdim}(X, A;\mathcal{S}_m^2)\leq n)$ if any uniformly continuous mapping $f:(X, A)\to(P, Q)\in\mathcal{S}_m^2$ is uniformly homotopic to one whose image is contained in $P^{(n)}\cup Q$.

Let (X, A) be a pair of uniform spaces, and let $\mathcal{U}$ and $\tilde{\mathcal{U}}$ be finite uniform coverings of X such that $\tilde{\mathcal{U}}$ refines $\mathcal{U}$. By $K(\mathcal{U})$ we denote the nerve of $\mathcal{U}$ and by $|K(\mathcal{U})|$ its geometric realization. We consider $|K(\mathcal{U}|A)|$ as a subpolyhedron of $|K(\mathcal{U})|$ embedded in the standard way; here $\mathcal{U}|A=\{U\cap A \mid U\in\mathcal{U}\}$. By

$$p_{\mathcal{U},\tilde{\mathcal{U}}}: (|K(\tilde{\mathcal{U}})|, |K(\tilde{\mathcal{U}}|A)|)\to(|K(\mathcal{U})|, |K(\mathcal{U}|A)|)$$

we denote a projection map.

We say that a *pair* (X, A) *of uniform spaces is f-movable* if for any finite uniform covering $\mathcal{U}_1$ of X there exists a finite uniform covering $\mathcal{U}_2$ which refines $\mathcal{U}_1$ and such that for any finite uniform covering $\mathcal{U}_3$ which refines $\mathcal{U}_1$ there exists a mapping

$$q:(|K(\mathcal{U}_2)|, |K(\mathcal{U}_2|A)|) \to (|K(\mathcal{U}_3)|, |K(\mathcal{U}_3|A)|)$$

such that the mappings $p_{\mathcal{U}_1,\mathcal{U}_3} \circ q$ and $p_{\mathcal{U}_1,\mathcal{U}_2}$ are homotopic. We say that a uniform space is f-movable if the pair $(X, \emptyset)$ is f-movable.

By $H_f^m(X, A; G)$ we denote the mth Čech cohomology group of a pair (X, A) of uniform spaces with coefficients in a group G, based on finite uniform coverings of X.

Observe that if X is a compact (Hausdorff) space or, respectively, (X, A) is a pair of compact spaces, then the above definitions of deformation dimension, movability and cohomology groups coincide with the standard ones.

For every uniform space X let pX be its precompact reflection; i.e., the uniform space with the same points as X and such that the finite coverings of X form a basis of uniform coverings for pX. The completion X^* of pX is a compact reflection of X and is called the Samuel compactification of X (see [11, p. 23]). If A is a subspace of the uniform space X then the closure of A in X^* can be considered as the Samuel compactification A^* of A. The reader should convince himself about the fact that any finite uniform covering $\mathcal{U}$ of $X \times I$ has a refinement of the form $\mathcal{V} \times \mathcal{W}$, where $\mathcal{V}$ and $\mathcal{W}$ are *finite* uniform coverings of X respectively I. Consequently every homotopy $X \times I \to Y$ with compact range space Y factors over $X^* \times I$. This property does not hold for the semi-uniform product $X * I$.

Since X is dense in X^* for any open covering $\mathcal{U}$ of X^* ($\mathcal{U}$ is uniform by [11, Theorem 23]) there is a canonical isomorphism of nerves $K(\mathcal{U}) \to K(\mathcal{U}|X)$ given by $\mathcal{U} \to \mathcal{U} \cap X$ for each $U \in \mathcal{U}$.

It is known [11, Theorem 10, p. 18] that if A is a subspace of a uniform space X and f a uniformly continuous mapping from A into a complete uniform space, then there is a unique uniformly continuous extension of f over the closure of A.

By using the above properties one can easily show the following

Proposition 3.1. *Let (X, A) be a pair of uniform spaces. Then*:

(i) *The deformation dimension* $\operatorname{defdim}(X, A; S_m^2)$ *of a pair of uniform spaces is equal to the (standard) deformation dimension* $\operatorname{defdim}(X^*, A^*; S_m^2)$.

(ii) *(X, A) is f-movable (as a pair of uniform spaces) if and only if the pair (X^*, A^*) of compact spaces is movable.*

(iii) $H_f^m(X, A; G) = H^m(X^*, A^*; G)$.

(iv) *Let P be a compact space and let $f^*: A^* \to P$ be the unique extension of a uniformly continuous map $f: A \to P$. Then f is uniformly extendable over X if and only if f^* is (continuously) extendable over X^*.*

(v) *Let f and g be uniformly continuous maps of X into a compact space P, and let f^* and g^*, respectively, be their extensions over X^*. Then f and g are uniformly homotopic if and only if f^* and g^* are homotopic.*

If P is an $(n-1)$-connected polyhedron, $n \geq 2$, then by l we denote the characteristic element of P; $l \in H^n(P; \pi_n(P))$. By using [17, Theorems 3.2 and 4.2] and the above proposition we obtain the following

Theorem 3.2. *Assume that $n \geq 2$, P is an $(n-1)$-connected compact ANR and (X, A) is a pair of uniform spaces such that $H_f^m(X, A; \pi_{m-1}(P)) = 0$ for every $m \geq n+2$ and which satisfies one of the following conditions:*

(i) *(X, A) is f-movable,*

(ii) *X is f-movable and* $\operatorname{defdim}(A; \mathcal{S}_{n-1}) < \infty$ *or*

(iii) $\operatorname{defdim}(X, A; \mathcal{S}^2_{n-1}) < \infty$.

Then a uniformly continuous map $f: A \to P$ is extendable over X if and only if $H^n(f)(l)$ is extendable over X.

Theorem 3.3. *Assume that $n \geq 2$, P is an $(n-1)$-connected ANR and X is a uniform space such that $H_f^m(X; \pi_m(P)) = 0 = H_f^{m+1}(X; \pi_m(P))$ for $m \geq n+1$. Suppose also that one of the following conditions is satisfied:*

(i) *X is f-movable,*

(ii) $\operatorname{defdim}(X; \mathcal{S}_{n-1}) < \infty$.

Then the set of uniform homotopy classes of the uniform maps from X to P is in one-to-one correspondence with the group $H_f^n(X; \pi_n(P))$ under the map $[f] \to H^n(f)(l)$.

Similarly we have

Theorem 3.4. *Assume that $n \geq 2$, P is an $(n-1)$-connected ANR and X is a uniform space such that $H_f^m(X; \pi_m(P)) = 0$ for $m \geq n+1$. Suppose also that one of the following conditions is satisfied:*

(i) *X is f-movable,*

(ii) $\operatorname{defdim}(X; \mathcal{S}_{n-1}) < \infty$.

Then two uniformly continuous mappings f and g from X to P are uniformly homotopic if and only if $H^n(f)(l) = H^n(g)(l)$.

The above theorems generalize some results in [13], which were proved directly by introducing an obstruction theory for uniform spaces. One can also observe that the above theorems can be stated in more general categories, e.g. the category of nearness spaces [18]. Thus one can also obtain a generalization of [18, Theorem 7.3.1].

Example. We want to show that the assumption of P being compact cannot be omitted from the above theorems. To this end we take the real line with its standard uniform structure as P and as X; since P is a 1-dimensional polyhedron it is an ANRU-space. Furthermore P is topologically contractible and hence n-connected for every n. As $A \subset X$ we use the subset of all natural numbers; A inherits the discrete uniform structure. One can verify that X and (X, A) are movable and hence the assumptions of Theorems 3.2, 3.3 and 3.4 are satisfied. Theorem 3.2 fails, because the map $f: A \to P$, $f(n) := 2^n$ does not have a *uniformly* continuous extension over X although $H^n(f)(l)$ is trivial. Theorems 3.3 and 3.4 fail because the identity mapping $f = \mathrm{id}: X \to P$ and a constant mapping $g: X \to P$ satisfy $H^n(g)(l) = H^n(g)(l) = 0$ although they are not *uniformly* homotopic.

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